

In blessed memory of  
Prof. Yu.F. Smirnov  
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# Extremal projectors for contragredient Lie (super)symmetries (short review)

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## Abstract

A brief review of the extremal projectors for contragredient Lie (super)-symmetries (finite-dimensional simple Lie algebras, basic classical Lie superalgebras, infinite-dimensional affine Kac-Moody algebras and superalgebras, as well as their quantum  $q$ -analogs) is given. Some bibliographic comments on the applications of extremal projectors are presented.

## 1 Introduction

Let  $G$  be a finite (or compact) group, and  $T$  be its unitary representation in a linear space  $V$  – that is, we have  $g \mapsto T(g)$  ( $g \in G$ ), where  $T(g)$  are linear operators acting in  $V$  and satisfying the condition  $T(g_1 g_2) = T(g_1)T(g_2)$ . The representation  $T$  in  $V$  is referred to as an irreducible one if  $\text{Lin}\{T(G)v\} = V$  for any nonzero vector  $v \in V$ . An irreducible representation (IR) is labeled with an additional superscript  $\lambda$ ,  $T^\lambda(g)$  (correspondingly,  $V^\lambda$ ) or, in matrix form,  $T^\lambda(g) = (t_{ij}^\lambda(g))$  ( $i, j = 1, 2, \dots, n$ ), where  $n$  is the dimension of  $V^\lambda$ .

It is well-known that the elements

$$P_{ij}^\lambda = \frac{n}{\dim G} \sum_{g \in G} T(g) t_{ij}^\lambda(g) \quad (1.1)$$

are projection operators for the finite group  $G$ ; that is, they satisfy the following properties:

$$P_{ij}^\lambda P_{kl}^{\lambda'} = \delta_{\lambda\lambda'} \delta_{jk} P_{il}^\lambda, \quad (P_{ij}^\lambda)^* = P_{ji}^\lambda, \quad (1.2)$$

where " \* " is Hermitian conjugation.

If  $G$  is a compact group then the formula (1.1) is modified as follows:

$$P_{ij}^\lambda = \frac{\dim \lambda}{|G|} \int_{g \in G} T(g) t_{ij}^\lambda(g) dg, \quad (1.3)$$

where  $|G|$  is the volume of the group  $G$  and  $dg$  is a  $g$ -invariant measure on the group  $G$ . In the case of  $G = SO(3)$  (or  $SU(2)$ ) we have

$$P_{mm'}^j = \frac{2j+1}{8\pi^2} \int T(\alpha, \beta, \gamma) D_{mm'}^j(\alpha, \beta, \gamma) \sin \beta d\alpha d\beta d\gamma, \quad (1.4)$$

where  $\alpha, \beta, \gamma$  are the Euler angles and  $D_{mm'}^j(\alpha, \beta, \gamma)$  is the Wigner  $D$ -function. The projection operator  $P^j := P_{jj}^j$  is referred to as the operator of projection onto the highest weight  $j$ .

Thus, we see that the projection operators in the form (1.1) or (1.3) require knowledge of explicit expressions for the operator function  $T(g)$  and the matrix elements of irreducible representations,  $t_{ij}^\lambda(g)$ , as well as (in the case of a compact group) the  $g$ -invariant measure  $dg$ . In the case of an arbitrary compact group  $G$ , the derivation of these expressions involves some problems. There naturally arises the question of whether it is possible to construct projection operators not in terms of the elements of a compact group and its representations but in the terms of its Lie algebra, since the compact group is completely determined by its Lie algebra. The answer to this question appears to be positive, and the history of its derivation is rather instructive. We briefly remind this history. It opens with the angular momentum Lie algebra.

The angular momentum Lie algebra  $\mathfrak{so}(3)$  ( $\simeq \mathfrak{su}(2)$ ) is generated by the three elements (generators)  $J_+$ ,  $J_-$  and  $J_0$  with the defining relations:

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_0, \quad J_\pm^* = J_\mp, \quad J_0^* = J_0. \quad (1.5)$$

It is obvious that the projector  $P^j = P_{jj}^j$ , (1.4), onto the highest weight  $j$  satisfies the relations:

$$J_+ P^j = P^j J_- = 0, \quad (P^j)^2 = P^j. \quad (1.6)$$

An associative polynomial algebra of the generators  $J_\pm$ ,  $J_0$  is called the universal enveloping algebra of the angular momentum Lie algebra and it is denoted by  $U(\mathfrak{so}(3))$  (or  $U(\mathfrak{su}(2))$ ).

The following proposition holds (no-go theorem): *No nontrivial solution of the set of the equations*

$$J_+ P = P J_- = 0 \quad (1.7)$$

*exists in the space  $U(\mathfrak{su}(2))$ ; that is, the unique solution of this set of the equations for  $P \in U(\mathfrak{su}(2))$  is a trivial one,  $P \equiv 0$ <sup>1</sup>.*

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<sup>1</sup> A rigorous mathematical formulation of this statement reads as follows: *The universal enveloping algebra  $U(\mathfrak{g})$  of a simple Lie algebra  $\mathfrak{g}$  has no zero divisors.*

Thus, the theorem states that the projector  $P^j$  does not exist in the form of a polynomial in the generators  $J_{\pm}$ ,  $J_0$ . This no-go theorem was well known to mathematicians, but we can assume that it was not known to the majority of physicists.

In 1964 (more than 45 years ago), the Swedish physicist and chemist P.-O. Löwdin<sup>2</sup>, who probably did not know the no-go theorem, published a paper in *Reviews Modern Physics* [1], where he considered the following operator:

$$P^j := \prod_{j' \neq j} \frac{\mathbf{J}^2 - j'(j'+1)}{j(j+1) - j'(j'+1)}. \quad (1.8)$$

Here  $\mathbf{J}^2$  is the Casimir element of the angular momentum Lie algebra:

$$\mathbf{J}^2 = \frac{1}{2} (J_+ J_- + J_- J_+) + J_0^2 = J_- J_+ + J_0 (J_0 + 1). \quad (1.9)$$

The operator (1.8) satisfies the equations

$$[J_0, P^j] = 0, \quad J_+ P^j = P^j J_- = 0, \quad (P^j)^2 = P^j \quad (1.10)$$

under the condition that the left- and right-hand sides of these equalities are applied to vectors characterized by the angular-momentum projection  $j$ ,  $J_0 \Psi_j = j \Psi_j$ . Therefore, the element (1.8) is the operator of projection onto the highest weight  $j$ .

After quite involved complicated calculations, Löwdin reduced the operator (1.8) to the form

$$P^j = \sum_{n \geq 0} \frac{(-1)^n (2j+1)!}{n! (2j+1+n)!} J_-^n J_+^n. \quad (1.11)$$

One year later, in 1965, another physicist, J. Shapiro from United States of America, published an article in *Journal of Mathematical Physics* [2], where he proposed the following: "Let us forget the initial expression in the form of infinite product, (1.8), and consider the defining equations (1.10), where  $P^j$  has the following ansatz:

$$P^j = \sum_{n \geq 0} C_n(j) J_-^n J_+^n. \quad (1.12)$$

Substituting this expression into (1.10), we immediately obtain the formula (1.11).

It is convenient to get rid of the superscript  $j$  in  $P^j$  by making the substitution  $j \rightarrow J_0$ . As a result, we arrive at

$$P = \sum_{n \geq 0} \frac{(-1)^n}{n!} \varphi_n(J_0) J_-^n J_+^n, \quad (1.13)$$

where

$$\varphi_n(J_0) = \prod_{k=1}^n (2J_0 + 1 + k)^{-1}. \quad (1.14)$$

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<sup>2</sup> Per-Olov Löwdin was born in 1916 in Uppsala, Sweden, and died in 2000 (see <http://www.quantum-chemistry-history.com/Lowdin1.htm>).

The element  $P$  is called the extremal projector. Acting by the extremal projector  $P$  on any weight  $\mathfrak{su}(2)$ -module  $M$  we obtain a space  $M^0 = pM$  of highest weight vectors for  $M$  (if  $pM$  has no singularities).

The extremal projector  $P$  does not belong to  $U(\mathfrak{su}(2))$ , but it belongs to some extension of the universal enveloping algebra. This extension is defined as follows.

Let us consider the formal Taylor series

$$\sum_{n,k \geq 0} C_{n,k}(J_0) J_-^n J_+^k \quad (1.15)$$

under the condition that  $|n - k| \leq N$  for some  $N \in \mathbb{Z}_+$ . The coefficients  $C_{n,k}(J_0)$  are rational functions of the generator  $J_0$ .

Let  $TU(\mathfrak{su}(2))$  be a linear space of such formal series. One can show that  $TU(\mathfrak{su}(2))$  is an associative algebra with respect to the multiplication of formal series. The associative algebra  $TU(\mathfrak{su}(2))$  is called the *Taylor extension* of  $U(\mathfrak{su}(2))$ . It is obvious that  $TU(\mathfrak{su}(2))$  contains  $U(\mathfrak{su}(2))$ .

The extremal projector (1.13) belongs to the Taylor extension  $TU(\mathfrak{su}(2))$ . Thus, Löwdin and Shapiro found a solution of the equations (1.10) in the extension of the space  $U(\mathfrak{su}(2))$  - that is, in  $TU(\mathfrak{su}(2))$  -rather than in the space  $U(\mathfrak{su}(2))$  itself.

Later, Shapiro tried to generalized the formula (1.13) to the case of  $\mathfrak{su}(3)$  ( $\mathfrak{u}(3)$ ). The Lie algebra  $\mathfrak{u}(3)$  is generated by nine elements  $e_{ik}$  ( $i, k = 1, 2, 3$ ) with the relations:

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}, \quad e_{ij}^* = e_{ij}. \quad (1.16)$$

Shapiro considered an ansatz for  $P := P(\mathfrak{su}(3))$  in the form

$$P := \sum_{\{n_i\}, \{m_i\}} C_{\{n_i\}, \{m_i\}}(e_{11}, e_{22}, e_{33}) e_{21}^{n_1} e_{31}^{n_2} e_{32}^{n_3} e_{12}^{m_1} e_{13}^{m_2} e_{23}^{m_3} \quad (1.17)$$

and used the equations

$$e_{ij}P = Pe_{ji} = 0 \quad (i < j), \quad [e_{ii}, P] = 0 \quad (i = 1, 2, 3). \quad (1.18)$$

From the last equations, it follows that

$$n_1 + n_2 - m_1 - m_2 = 0, \quad n_2 + n_3 - m_2 - m_3 = 0. \quad (1.19)$$

Under the conditions (1.19) the expression (1.17) belongs to  $TU(\mathfrak{su}(3))$ . A set of equations for the coefficients  $C_{\{n_i\}, \{m_i\}}(e_{11}, e_{22}, e_{33})$  proved to be rather complicated, and Shapiro failed to solve it and stopped at this stage.

In 1968 R.M. Asherova and Yu.F. Smirnov [3] made the first important step towards deriving an explicit formula for the extremal projector for  $\mathfrak{u}(3)$ . They proposed applying the Shapiro ansatz (1.17) to the extremal projector for the  $\mathfrak{su}(2)$  ( $T$ -spin) subalgebra generated by the elements  $e_{23}$ ,  $e_{32}$  and  $e_{22} - e_{33}$ ; that is

$$P_{23} = \sum_{n \geq 0} \frac{(-1)^n}{n!} \varphi_n(e_{22} - e_{33}) e_{32}^n e_{23}^n, \quad (1.20)$$

where

$$\varphi_n(e_{22} - e_{33}) = \prod_{k=1}^n (e_{22} - e_{33} + 1 + k)^{-1}. \quad (1.21)$$

Since  $e_{23}P_{23} = 0$ , the operator  $P$  takes the form

$$P = \sum_{n_i \geq 0} C_{n_1, n_2, n_3}(e_{11}, e_{22}, e_{33}) e_{21}^{n_1} e_{31}^{n_2} e_{32}^{n_3} e_{12}^{n_1 - n_3} e_{13}^{n_2 + n_3} P_{23}. \quad (1.22)$$

In this case, the set of equations for the coefficients  $C_{\{n_i\}}(\{e_{ii}\})$  becomes simpler than the corresponding set of equations for the Shapiro ansatz (1.17). It was solved, although the resulting expressions for the coefficients  $C_{\{n_i\}}(\{e_{ii}\})$  are rather complicated.

The next simple idea [4] was to apply, to the expression (1.22) from the left, the extremal projector of the  $\mathfrak{su}(2)$  subalgebra generated by  $e_{12}$ ,  $e_{21}$  and  $e_{11} - e_{22}$ . As a result, we obtain the extremal projector  $P(\mathfrak{su}(3))$  in the simple form

$$P(\mathfrak{su}(3)) = P_{12} \left( \sum_{n \geq 0} C_n(e_{11} - e_{33}) e_{31}^n e_{13}^n \right) P_{23}. \quad (1.23)$$

The final expression is given by

$$P(\mathfrak{su}(3)) = P_{12} P_{13} P_{23}, \quad (1.24)$$

where

$$P_{ij} = \sum_{n \geq 0} \frac{(-1)^n}{n!} \varphi_{ij,n}(e_{ii} - e_{jj}) e_{ji}^n e_{ij}^n \quad (i < j), \quad (1.25)$$

$$\varphi_{ij,n}(e_{ii} - e_{jj}) = \prod_{k=1}^n (e_{ii} - e_{jj} + j - i + k)^{-1}. \quad (1.26)$$

It turned out this formula is a key one. We rewrite this expression for  $P(\mathfrak{su}(3))$  in the terms of the Cartan–Weyl basis with Greek indexes, namely we replace the root indexes 12, 23, 13 by  $\alpha$ ,  $\beta$ ,  $\alpha + \beta$  correspondingly. Moreover we set  $h_\alpha := e_{11} - e_{22}$ ,  $h_\beta := e_{22} - e_{33}$ ,  $h_{\alpha+\beta} := e_{11} - e_{33}$ . In these terms the extremal projector  $P(\mathfrak{su}(3))$  has the form

$$P(\mathfrak{su}(3)) = P_\alpha P_{\alpha+\beta} P_\beta, \quad (1.27)$$

where

$$P_\gamma = \sum_{n \geq 0} \frac{(-1)^n}{n!} \varphi_{\gamma,n} e_{-\gamma}^n e_\gamma^n, \quad (1.28)$$

$$\varphi_{\gamma,n} = \prod_{k=1}^n \left( h_\gamma + \rho(\gamma) + \frac{1}{2}(\gamma, \gamma)k \right)^{-1}. \quad (1.29)$$

Here  $\rho$  is a linear function on the positive root system  $\Delta_+ = \{\alpha, \beta, \alpha + \beta\}$ , such that  $\rho(\pi) = \frac{1}{2}(\pi, \pi)$  for all simple roots  $\pi \in \Pi := \{\alpha, \beta\}$ .

Later, the explicit formulas (1.27)–(1.29) were generalized to all finite-dimensional simple Lie algebras [5]–[7], basic classical Lie superalgebras [8], infinite-dimensional affine Kac-Moody algebras and superalgebras [9], as well as all their quantum  $q$ -analogs [10, 11]. At the present time, the extremal projector method – that uses explicit formulas of the extremal projectors – is a powerful and universal method for solving many problems in the representation theory, which are widely applied in the theoretical and mathematical physics. For instance, the method makes it possible to classify irreducible modules, to decompose them into submodules (for example, to analyze the structure of Verma modules), to describe reduced (super)algebras (which are associated with the reduction of a (super)algebra to a subalgebra), to construct bases of representations (for example, the Gelfand-Tsetlin's type), to develop a detailed theory of Clebsch-Gordan coefficients and other elements of Wigner-Racah calculus (including compact analytic formulas for these elements and their symmetry properties), and so on.

The ensuing exposition is organized as follows. In Section 2 we describe the extremal projectors for finite-dimensional simple Lie algebras. In Section 3 we present the extremal projectors for finite-dimensional basic classical Lie superalgebras. In Section 4 the extremal projectors for infinite-dimensional affine Kac-Moody algebras and superalgebras are considered. In Section 5 we describe the extremal projectors for quantum  $q$ -analogs of Lie algebras and superalgebras. Finally, in Section 6 some bibliographic comments on the applications of extremal projectors are presented.

It should be noted that a short review of the extremal projector method was already published in [12], however there we considered explicit formulas of the extremal projectors only for simple Lie algebras and we demonstrated some applications of these projectors for derivation of the general analytic formulas of Clebsch-Gordan coefficients for the  $su(2)$  and  $su(3)$  Lie algebras.

## 2 Extremal projectors for simple Lie algebras

Let  $\mathfrak{g}$  be a finite-dimensional simple Lie algebra of rank  $r$  and  $\Pi := \{\alpha_1, \alpha_2, \dots, \alpha_r\}$  be its simple root system. Let  $\Delta_+(\mathfrak{g})$  be a system of all positive roots of  $\mathfrak{g}$ . Any root  $\gamma$  of  $\Delta_+(\mathfrak{g})$  has the form:  $\gamma = \sum_i^r l_i^{(\gamma)} \alpha_i$ , where all  $l_i^{(\gamma)}$  are nonnegative integers. A generalization of the formulas (1.27)–(1.29) to the case of arbitrary finite-dimensional simple Lie algebra  $\mathfrak{g}$  is associated with the concept of normal ordering in the system  $\Delta_+(\mathfrak{g})$ .

**Definition 2.1** *We say that the system  $\Delta_+(\mathfrak{g})$  is in normal (convex) ordering if each composite (not simple) root  $\gamma = \alpha + \beta$  ( $\alpha, \beta, \gamma \in \Delta_+(\mathfrak{g})$ ) is written between its components  $\alpha$  and  $\beta$ . It means that in the normal ordering system  $\Delta_+(\mathfrak{g})$  we have either*

$$\dots, \alpha, \dots, \alpha + \beta, \dots, \beta \dots, \quad (2.1)$$

or

$$\dots, \beta, \dots, \alpha + \beta, \dots, \alpha, \dots \quad (2.2)$$

We say also that  $\alpha \prec \beta$  if  $\alpha$  is located on the left side of  $\beta$  in the normal ordering system  $\Delta_+(\mathfrak{g})$ , i.e. this corresponds to the case (2.1).

The normally ordered system  $\Delta_+(\mathfrak{g})$  is denoted by the symbol  $\vec{\Delta}_+(\mathfrak{g})$ . It is evident that two boundary (end) roots in  $\vec{\Delta}_+(\mathfrak{g})$  are simple.

Let us write down the normal orderings for all simple Lie algebras of rank 2 (see [7]):

$$A_1 \otimes A_1 : \alpha, \beta \leftrightarrow \beta, \alpha, \quad (2.3)$$

$$A_2 : \alpha, \alpha + \beta, \beta \leftrightarrow \beta, \alpha + \beta, \alpha, \quad (2.4)$$

$$B_2 : \alpha, \alpha + \beta, \alpha + 2\beta, \beta \leftrightarrow \beta, \alpha + 2\beta, \alpha + \beta, \alpha, \quad (2.5)$$

$$G_2 : \alpha, \alpha + \beta, 2\alpha + 3\beta, \alpha + 2\beta, \alpha + 3\beta, \beta \leftrightarrow \beta, \alpha + 3\beta, \alpha + 2\beta, 2\alpha + 3\beta, \alpha + \beta, \alpha, \quad (2.6)$$

where  $\alpha - \beta$  is not any root.

We say that a positive root  $\gamma \in \Delta_+(\mathfrak{g})$  is generated by positive roots  $\alpha$  and  $\beta$  if it is presented in the form of their linear combination:  $\gamma = k\alpha + l\beta$ , where  $k, l$  are nonzero integers.

Let  $\alpha$  and  $\beta$  be any two roots from  $\Delta_+(\mathfrak{g})$ . We denote by  $\{\alpha, \beta\}$  the subset which contains the roots  $\alpha, \beta$  and all positive roots from  $\Delta_+(\mathfrak{g})$ , generated by the roots  $\alpha$  and  $\beta$ .

**Definition 2.2** Let in the normal ordering system  $\vec{\Delta}_+(\mathfrak{g})$  between two roots from the subset  $\{\alpha, \beta\}$  there are not another roots except the roots generated by themselves  $\alpha$  and  $\beta$ . Then inverting this subset we again obtain the normal ordering system  $\vec{\Delta}'_+(\mathfrak{g})$ . Such transformation is called the elementary inversion.

Because any two positive roots generate a root system of a Lie algebra of rank 2, therefore all elementary inversions coincide with the elementary inversions of normal orders for the root systems  $\Delta_+(\mathfrak{g})$  of the Lie algebras  $\mathfrak{g}$  of rank 2, (2.3)–(2.6).

The combinatorial structure of the root system  $\Delta_+(\mathfrak{g})$  is described the following theorem.

**Theorem 2.3** (i) Normal ordering in the system  $\Delta_+(\mathfrak{g})$  exists for any mutual location of the simple roots  $\alpha_i$ ,  $i = 1, 2, \dots, r$ . (ii) Any two normal orderings  $\vec{\Delta}_+(\mathfrak{g})$  and  $\vec{\Delta}'_+(\mathfrak{g})$  can be obtained one from another by compositions of elementary inversions for the root systems of the Lie algebras of rank 2.

A detailed proof of the theorem is presented in [7, 13].

Let  $e_{\pm\gamma}, h_\gamma$  be Cartan-Weyl root vectors normalized by the condition

$$[e_\gamma, e_{-\gamma}] = h_\gamma. \quad (2.7)$$

We construct a formal Taylor series on the following monomials

$$e_{-\beta}^{n_\beta} \cdots e_{-\gamma}^{n_\gamma} e_{-\alpha}^{n_\alpha} e_\alpha^{m_\alpha} e_\gamma^{m_\gamma} \cdots e_\beta^{m_\beta} \quad (2.8)$$

with coefficients which are rational functions of the Cartan elements  $h_{\alpha_i}$  ( $i = 1, 2, \dots, r$ ), and nonnegative integers  $n_\beta, \dots, n_\gamma, n_\alpha, m_\alpha, m_\gamma, \dots, m_\beta$  are subjected to the constraints (for some  $N \in \mathbb{Z}_+$ )

$$\left| \sum_{\gamma \in \Delta_+} (n_\gamma - m_\gamma) l_i^{(\gamma)} \right| \leq N, \quad i = 1, 2, \dots, r, \quad (2.9)$$

for all monomial of the given series. Here  $l_i^{(\gamma)}$  are coefficients in a decomposition of the root  $\gamma$  with respect to the system of simple roots  $\Pi$ ,  $\gamma = \sum_i^r l_i^{(\gamma)} \alpha_i$ ,  $\alpha_i \in \Pi$ . Let  $TU(\mathfrak{g})$  be a linear space of all such formal series. We have the following simple proposition.

**Proposition 2.4** *The linear space  $TU(\mathfrak{g})$  is an associative algebra with respect to a multiplication of formal series.*

The algebra  $TU(\mathfrak{g})$  is called the Taylor extension of  $U(\mathfrak{g})$ .

**Theorem 2.5** *The equations*

$$e_\gamma P(\mathfrak{g}) = P(\mathfrak{g}) e_{-\gamma} = 0 \quad (\forall \gamma \in \Delta_+(\mathfrak{g})) , \quad P^2(\mathfrak{g}) = P(\mathfrak{g}) \quad (2.10)$$

have a unique nonzero solution in the space  $TU(\mathfrak{g})$ ,  $P(\mathfrak{g}) \in TU(\mathfrak{g})$ , and this solution has the form

$$P(\mathfrak{g}) = \prod_{\gamma \in \vec{\Delta}_+(\mathfrak{g})} P_\gamma \quad (2.11)$$

for any normal ordering system  $\vec{\Delta}_+(\mathfrak{g})$ , where the elements  $P_\gamma$  are defined by the formulae

$$P_\gamma = \sum_{n \geq 0} \frac{(-1)^n}{n!} \varphi_{\gamma, n} e_{-\gamma}^n e_\gamma^n, \quad (2.12)$$

$$\varphi_{\gamma, n} = \prod_{k=1}^n \left( h_\gamma + \rho(\gamma) + \frac{1}{2}(\gamma, \gamma) k \right)^{-1}, \quad (2.13)$$

and  $\rho$  is the linear function on the positive root system  $\Delta_+(\mathfrak{g})$ , such that  $\rho(\alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$  for all simple roots  $\alpha_i \in \Pi^3$ . Thus, the extremal projector  $P(\mathfrak{g})$  does not depend on choice of normal order in  $\Delta_+(\mathfrak{g})$ .

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<sup>3</sup>In the case of the finite-dimensional simple Lie algebras, the function  $\rho(\cdot)$  can be presented in the form of the scalar product  $\rho(\gamma) = (\rho, \gamma)$ , where  $\rho$  on the right-hand side is a half-sum of all positive roots.

The extremal projectors of the Lie algebras of rank 2 and the combinatorial theorem play key roles for the proof of this theorem for an arbitrary simple Lie algebra  $\mathfrak{g}$  of rank  $r > 2$  [7, 13]. The extremal projectors of the Lie algebras of rank 2 are given by (see [7])

$$P_\alpha P_\beta = P_\beta P_\alpha \quad (A_1 \otimes A_1), \quad (2.14)$$

$$P_\alpha P_{\alpha+\beta} P_\beta = P_\beta P_{\alpha+\beta} P_\alpha \quad (A_2), \quad (2.15)$$

$$P_\alpha P_{\alpha+\beta} P_{\alpha+2\beta,\beta} = P_\beta P_{\alpha+2\beta} P_{\alpha+\beta} P_\alpha \quad (B_2), \quad (2.16)$$

$$P_\alpha P_{\alpha+\beta} P_{2\alpha+3\beta} P_{\alpha+2\beta} P_{\alpha+3\beta} P_\beta = P_\beta P_{\alpha+3\beta} P_{\alpha+2\beta} P_{2\alpha+3\beta} P_{\alpha+\beta} P_\alpha \quad (G_2). \quad (2.17)$$

We can show that these equations are valid not only for  $\rho(\alpha) = \frac{1}{2}(\alpha, \alpha)$  and  $\rho(\beta) = \frac{1}{2}(\beta, \beta)$  but as well as for  $\rho(\alpha) = x_\alpha$  and  $\rho(\beta) = x_\beta$  where  $x_\alpha$  and  $x_\beta$  are arbitrary complex numbers. Taking into account the second part (ii) of the theorem (2.3) it now follows the equalities (2.10) for (2.11)–(2.13) immediately. A uniqueness of this solution in the space  $TU(\mathfrak{g})$  is proved more complicated [14].

### 3 Extremal projectors for Lie superalgebras

In this section we consider the extremal projectors for finite-dimensional basic classical Lie superalgebras. Let  $\mathfrak{g}$  be a finite-dimensional basic classical Lie superalgebra [15, 16] of rank  $r$  and  $\Pi := \{\alpha_1, \alpha_2, \dots, \alpha_r\}$  be its simple root system. Let  $\Delta_+(\mathfrak{g})$  be a system of all positive roots of  $\mathfrak{g}$ . Any root  $\gamma$  of  $\Delta_+(\mathfrak{g})$  has the form:  $\gamma = \sum_i^r l_i^{(\gamma)} \alpha_i$ , where all  $l_i^{(\gamma)}$  are nonnegative integers. There are two type of the roots: even and odd,  $\Delta_+(\mathfrak{g}) = \Delta_+^{(0)}(\mathfrak{g}) \oplus \Delta_+^{(1)}(\mathfrak{g})$ , and they are classified (colored) as follows:

- Any even root  $\gamma \in \Delta_+(\mathfrak{g}^{(0)})$  is white. In this case  $2\gamma \notin \Delta_+^{(0)}(\mathfrak{g})$  and  $(\gamma, \gamma) \neq 0$ .
- A odd root  $\gamma \in \Delta_+^{(1)}(\mathfrak{g})$  is called grey if  $2\gamma \notin \Delta_+(\mathfrak{g}^{(0)})$ . In this case  $(\gamma, \gamma) = 0$ .
- A odd root  $\gamma \in \Delta_+^{(1)}(\mathfrak{g})$  is called dark if  $2\gamma \in \Delta_+^{(0)}(\mathfrak{g})$ . In this case  $(\gamma, \gamma) \neq 0$ .

The total system of all roots,  $\Delta(\mathfrak{g})$ , has the form:  $\Delta(\mathfrak{g}) = \Delta(\mathfrak{g})_+ \cup (-\Delta_+(\mathfrak{g}))$  and the parity of the negative root  $-\gamma$  coincides with the parity of the positive root  $\gamma$ .

Now we define of a reduced system of the positive root system  $\Delta_+(\mathfrak{g})$ .

**Definition 3.1** *The reduced system is called a set  $\underline{\Delta}_+(\mathfrak{g})$  obtained from the positive system  $\Delta_+(\mathfrak{g})$  by removing of all doubled roots  $2\gamma$  where  $\gamma$  is a dark odd root, that is  $\underline{\Delta}_+(\mathfrak{g}) = \Delta_+(\mathfrak{g}) \setminus \{2\gamma \in \Delta_+(\mathfrak{g}) \mid \gamma \text{ is odd}\}$ .<sup>4</sup>*

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<sup>4</sup>In the case of any simple Lie algebra  $\mathfrak{g}$ , the reduced system coincides with the total positive root system,  $\underline{\Delta}_+(\mathfrak{g}) = \Delta_+(\mathfrak{g})$ .

Normal ordering in the system  $\underline{\Delta}_+(\mathfrak{g})$  is fixed by the definition (2.1). The combinatorial theorem (2.3) is also true for the reduced system  $\underline{\Delta}_+(\mathfrak{g})$ . In this case there are the following elementary inversions of the reduced systems of the Lie algebras and superalgebras of rank 2 (see [17]):

$$\alpha, \beta \leftrightarrow \beta, \alpha, \quad (3.1)$$

$$\alpha, \alpha+\beta, \beta \leftrightarrow \beta, \alpha+\beta, \alpha, \quad (3.2)$$

$$\alpha, \alpha+\beta, \alpha+2\beta, \beta \leftrightarrow \beta, \alpha+2\beta, \alpha+\beta, \alpha, \quad (3.3)$$

$$\alpha, \alpha+\beta, 2\alpha+3\beta, \alpha+2\beta, \alpha+3\beta, \beta \leftrightarrow \beta, \alpha+3\beta, \alpha+2\beta, 2\alpha+3\beta, \alpha+\beta, \alpha, \quad (3.4)$$

where  $\alpha - \beta$  is not any root.<sup>5</sup>

The Cartan-Weyl root vectors  $e_{\pm\gamma}$ ,  $h_\gamma$  ( $\gamma \in \underline{\Delta}_+(\mathfrak{g})$ ) are normalized by the condition

$$[e_\gamma, e_{-\gamma}] = h_\gamma. \quad (3.5)$$

The Taylor extension  $TU(\mathfrak{g})$  of  $U(\mathfrak{g})$  is generated by the formal series on the monomials (2.8) where the roots  $\alpha, \gamma, \dots, \beta$  belong to  $\underline{\Delta}_+(\mathfrak{g})$ .

The extremal projector, namely the element  $P(\mathfrak{g})$  satisfying the conditions (2.10), is given by the formula [8]

$$P(\mathfrak{g}) = \prod_{\gamma \in \underline{\Delta}_+(\mathfrak{g})} P_\gamma, \quad (3.6)$$

where the factors  $P_\gamma$  are defined as follows. If the root  $\gamma$  is white, then

$$P_\gamma = \sum_{n \geq 0} \frac{(-1)^n}{n!} \varphi_{\gamma,n} e_{-\gamma}^n e_\gamma^n, \quad (3.7)$$

where

$$\varphi_{\gamma,n} = \prod_{k=1}^n \left( h_\gamma + \rho(\gamma) + \frac{1}{2}(\gamma, \gamma)k \right)^{-1}. \quad (3.8)$$

If the root  $\gamma$  is gray, then

$$P_\gamma = 1 - \frac{1}{h_\gamma + \rho(\gamma)} e_{-\gamma} e_\gamma. \quad (3.9)$$

If the root  $\gamma$  is dark, then the factors  $P_\gamma$  is given by (cf. [18] for the  $\mathfrak{osp}(1|2)$  case)

$$P_\gamma = \sum_{n \geq 0} \frac{1}{n!} \left( \varphi_{\gamma,2n}^{(0)} e_{-\gamma}^{2n} e_\gamma^{2n} + \varphi_{\gamma,2n+1}^{(1)} e_{-\gamma}^{2n+1} e_\gamma^{2n+1} \right), \quad (3.10)$$

where

$$\varphi_{\gamma,2n}^{(0)} = 2^{-n} \prod_{k=1}^n \left( h_\gamma + \rho(\gamma) + \frac{1}{2}(\gamma, \gamma)(k-1) \right)^{-1}, \quad (3.11)$$

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<sup>5</sup>Either of the two roots  $\alpha$  and  $\beta$  in (3.1) can be white, gray or dark. Either of the two roots  $\alpha, \beta$  in (3.2) and also the root  $\alpha$  in (3.3) is white or gray. The root  $\beta$  in (3.3) is white or dark. The roots  $\alpha$  and  $\beta$  in (3.4) are white.

$$\varphi_{\gamma,2n+1}^{(1)} = -2^{-n} \prod_{k=1}^{n+1} \left( h_\gamma + \rho(\gamma) + \frac{1}{2}(\gamma, \gamma)k \right)^{-1}. \quad (3.12)$$

The linear function  $\rho(\gamma)$ ,  $\rho(\alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$  for all simple roots  $\alpha_i \in \Pi$ , can be presented in the form of the scalar product  $\rho(\gamma) = (\rho, \gamma)$ , where  $\rho$  on the right-hand side is a difference between a half-sum of all even positive roots and a half-sum of all odd positive roots.

## 4 Extremal projectors for affine Kac–Moody (super)algebras

Let  $\mathfrak{g}$  be a affine Kac–Moody (super)algebra [19, 20] of rank  $r$  and  $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$  be its simple root system. Let  $\Delta_+(\mathfrak{g})$  be a system of all positive roots of  $\mathfrak{g}$ . Any root  $\gamma$  of  $\Delta_+(\mathfrak{g})$  has the form:  $\gamma = \sum_i^r l_i^{(\gamma)} \alpha_i$ , where all  $l_i^{(\gamma)}$  are nonnegative integers. In this case the system  $\Delta_+(\mathfrak{g}) = \Delta_+^{(0)}(\mathfrak{g}) \oplus \Delta_+^{(1)}(\mathfrak{g})$  has infinite number of the roots which are categorized into real and imaginary. Every imaginary root  $m\delta$  ( $m \in \mathbb{N}$ ) satisfies the condition  $(m\delta, \gamma) = 0$  for all  $\gamma \in \Delta(\mathfrak{g}) (= \Delta_+(\mathfrak{g}) \cup (-\Delta_+(\mathfrak{g})))$ . For the real roots this condition is not valid. We again fix the reduced system  $\underline{\Delta}_+(\mathfrak{g})$ , which is obtained from the total system  $\Delta_+(\mathfrak{g})$  by removing of all doubled roots  $2\gamma$  where  $\gamma$  is a dark odd root, and then we define the normal ordering in  $\underline{\Delta}_+(\mathfrak{g})$  as follows.

**Definition 4.1** *We say that the system  $\underline{\Delta}_+(\mathfrak{g})$  is in normal ordering if its roots are written as follows: (i) all imaginary roots are immediately adjacent to one another, (ii) each composite (not simple) root  $\gamma = \alpha + \beta$  ( $\alpha, \beta, \gamma \in \underline{\Delta}_+(\mathfrak{g})$ ), where  $\alpha$  and  $\beta$  are not proportional roots ( $\alpha \neq \lambda\beta$ ), is written between its components  $\alpha$  and  $\beta$ .*

The combinatorial theorem (2.3) is also true for the reduced system  $\underline{\Delta}_+(\mathfrak{g})$ . In this case there are the following elementary inversions of the reduced systems of algebras and superalgebras of rank 2 (see [13, 21]):

$$\alpha, \beta \leftrightarrow \beta, \alpha, \quad (4.1)$$

$$\alpha, \alpha + \beta, \beta \leftrightarrow \beta, \alpha + \beta, \alpha, \quad (4.2)$$

$$\alpha, \alpha + \beta, \alpha + 2\beta, \beta \leftrightarrow \beta, \alpha + 2\beta, \alpha + \beta, \alpha. \quad (4.3)$$

$$\alpha, \alpha + \beta, 2\alpha + 3\beta, \alpha + 2\beta, \alpha + 3\beta, \beta \leftrightarrow \beta, \alpha + 3\beta, \alpha + 2\beta, 2\alpha + 3\beta, \alpha + \beta, \alpha. \quad (4.4)$$

$$\begin{aligned} \alpha, \delta + \alpha, 2\delta + \alpha, \dots, \infty\delta + \alpha, \delta, 2\delta, 3\delta, \dots, \infty\delta, \infty\delta - \alpha, \dots, 2\delta - \alpha, \delta - \alpha &\leftrightarrow \\ \leftrightarrow \delta - \alpha, 2\delta - \alpha, \dots, \infty\delta - \alpha, \delta, 2\delta, 3\delta, \dots, \infty\delta, \infty\delta + \alpha, \dots, 2\delta + \alpha, \delta + \alpha, \alpha, \end{aligned} \quad (4.5)$$

$$\begin{aligned} \alpha, \delta + 2\alpha, \delta + \alpha, 3\delta + 2\alpha, 2\delta + \alpha, \dots, \infty\delta + \alpha, (2\infty + 1)\delta + 2\alpha, (\infty + 1)\delta + \alpha, \delta, 2\delta, \dots, \\ \infty\delta, (\infty + 1)\delta - \alpha, (2\infty + 1)\delta - 2\alpha, \infty\delta - \alpha, \dots, 2\delta - \alpha, 3\delta - 2\alpha, \delta - \alpha, \delta - 2\alpha, \leftrightarrow \\ \leftrightarrow \delta - 2\alpha, \delta - \alpha, 3\delta - 2\alpha, 2\delta - \alpha, \dots, \infty\delta - \alpha, (2\infty + 1)\delta - 2\alpha, (\infty + 1)\delta - \alpha, \delta, 2\delta, \dots, \\ \infty\delta, (\infty + 1)\delta + \alpha, (2\infty + 1)\delta + 2\alpha, \infty\delta + \alpha, \dots, 2\delta + \alpha, 3\delta + 2\alpha, \delta + \alpha, \delta + 2\alpha, \alpha, \end{aligned} \quad (4.6)$$

where  $\alpha - \beta$  is not any root.

The Cartan-Weyl root vectors  $e_{\pm\gamma}$ ,  $h_\gamma$  ( $\gamma \in \underline{\Delta}_+(\mathfrak{g})$ ) are normalized by the condition<sup>6</sup>

$$[e_\gamma, e_{-\gamma}] = h_\gamma. \quad (4.7)$$

We construct a formal Taylor series on the following monomials

$$e_{-\beta}^{n_\beta} \cdots e_{-\gamma}^{n_\gamma} e_{-\alpha}^{n_\alpha} e_{\alpha'}^{m_{\alpha'}} e_{\gamma'}^{m_{\gamma'}} \cdots e_{\beta'}^{m_{\beta'}} \quad (\text{finite product}) \quad (4.8)$$

with coefficients which are rational functions of the Cartan elements  $h_{\alpha_i}$  ( $i = 1, 2, \dots, r$ ), and nonnegative integers  $n_\beta, \dots, n_\gamma, n_\alpha, m_{\alpha'}, m_{\gamma'}, \dots, m_{\beta'}$  are subjected to the constraints

$$\left| \sum_{\gamma \in \underline{\Delta}_+} n_\gamma l_i^{(\gamma)} - \sum_{\gamma' \in \underline{\Delta}_+} m_{\gamma'} l_i^{(\gamma')} \right| \leq N, \quad i = 1, 2, \dots, r, \quad (4.9)$$

for all monomial of the given series. Here  $l_i^{(\gamma)}$  are coefficients in a decomposition of the root  $\gamma$  with respect to the system of simple roots  $\Pi$ . Let  $TU(\mathfrak{g})$  be a linear space of all such formal series. *The linear space  $TU(\mathfrak{g})$  is an associative algebra with respect to a multiplication of formal series.* The algebra  $TU(\mathfrak{g})$  is called the Taylor extension of  $U(\mathfrak{g})$ .

The extremal projector  $P(\mathfrak{g})$  of the affine Kac-Moody (super)algebra is given by the formula [9]:

$$P(\mathfrak{g}) = \prod_{\gamma \in \underline{\Delta}_+(\mathfrak{g})} P_\gamma, \quad (4.10)$$

where the factors  $P_\gamma$  are defined by the formulas (3.7)–(3.12) for all real roots. If the root  $\gamma$  is even imaginary,  $\gamma = m\delta$  ( $m \in \mathbb{N}$ ), then<sup>7</sup>

$$P_{m\delta} = \prod_{i=0}^{\dim \mathfrak{g}_{m\delta}} P_{m\delta}^{(i)} \quad (4.11)$$

$$P_{m\delta}^{(i)} = \sum_{n \geq 0} \frac{(-1)^n}{n! (h_{m\delta} + \rho(m\delta))^n} e_{-m\delta, i}^n e_{m\delta, i}^n. \quad (4.12)$$

The linear function  $\rho(\gamma)$  is determined by  $\rho(\alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$  for all simple roots  $\alpha_i \in \Pi$ .

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<sup>6</sup>If a root space  $\mathfrak{g}_{m\delta}$  ( $\mathfrak{g}_{-m\delta}$ ) of the imaginary root  $m\delta$  ( $-m\delta$ ),  $m \in \mathbb{N}$ , is not multiplicity free,  $\dim \mathfrak{g}_{m\delta} = \dim \mathfrak{g}_{-m\delta} > 1$ , then we choose basis root vectors  $\{e_{-m\delta, i}\}$  and  $\{e_{m\delta, i}\}$  that are dual with respect to a standard bilinear form on  $\mathfrak{g}$ ,  $(e_{-m\delta, i}, e_{m\delta, j}) = \delta_{ij}$ . In this case  $[e_{m\delta, i}, e_{-m\delta, j}] = \delta_{ij} h_{m\delta}$ , and moreover  $[e_{\pm m\delta, i}, e_{\pm m\delta, j}] = 0$ .

<sup>7</sup>In the case of the superalgebra  $\mathfrak{g} = A^{(4)}(2k, 2l)$  the imaginary roots can be white (even) as well as dark (odd). If the root  $m\delta$  is a dark imaginary root, then the formula (4.12) is modified slightly by analogy with (3.10).

## 5 Extremal projectors for quantum Lie (super)-algebras

Let  $\mathfrak{g}(A, \tau)$  be a contragredient Lie (super)algebra of finite growth<sup>8</sup> with a symmetrizable Cartan matrix  $A$  (i.e.  $A = DA^{sym}$ , where  $A^{sym} = (a_{ij}^{sym})_{i,j \in I}$  is a symmetrical matrix, and  $D$  is an invertible diagonal matrix,  $D = \text{diag}(d_1, d_2, \dots, d_r)$ ),  $\tau \subset I$ ,  $I := \{1, 2, \dots, r\}$ , and let  $\Pi := \{\alpha_1, \dots, \alpha_r\}$  be a system of simple roots for  $\mathfrak{g}(A, \tau)$ .

**Definition 5.1** [10, 17] *The quantum (super)algebra  $U_q(\mathfrak{g})$  (where  $\mathfrak{g} := \mathfrak{g}(A, \tau)$ ) is an associative (super)algebra over  $\mathbb{C}[q, q^{-1}]$  with Chevalley generators  $e_{\pm\alpha_i}$ ,  $k_{\alpha_i}^{\pm 1} := q^{\pm h_{\alpha_i}}$ , ( $i \in I := \{1, 2, \dots, r\}$ ), and the defining relations:*

$$k_{\alpha_i} k_{\alpha_i}^{-1} = k_{\alpha_i}^{-1} k_{\alpha_i} = 1, \quad k_{\alpha_i} k_{\alpha_j} = k_{\alpha_j} k_{\alpha_i}, \quad (5.1)$$

$$k_{\alpha_i} e_{\pm\alpha_j} k_{\alpha_i}^{-1} = q^{\pm(\alpha_i, \alpha_j)} e_{\pm\alpha_j}, \quad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij} \frac{k_{\alpha_i} - k_{\alpha_i}^{-1}}{q - q^{-1}}, \quad (5.2)$$

$$(\text{ad}_q e_{\pm\alpha_i})^{n_{ij}} e_{\pm\alpha_j} = 0 \quad \text{for } i \neq j, \quad (5.3)$$

where the positive integers  $n_{ij}$  are given as follows:  $n_{ij} = 1$  if  $a_{ii}^{sym} = a_{ij}^{sym} = 0$ ,  $n_{ij} = 2$  if  $a_{ii}^{sym} = 0$ ,  $a_{ij}^{sym} \neq 0$ , and  $n_{ij} = -2a_{ij}^{sym}/a_{ii}^{sym} + 1$  if  $a_{ii}^{sym} \neq 0$ . Moreover, if any three simple roots  $\alpha_i, \alpha_j, \alpha_k$  satisfy the conditions  $(\alpha_i, \alpha_i) = (\alpha_j, \alpha_k) = 0$  and  $(\alpha_i, \alpha_j) = -(\alpha_i, \alpha_k) \neq 0$ , then there are the additional triple relations of the form<sup>9</sup>:

$$[[e_{\pm\alpha_i}, e_{\pm\alpha_j}]_q, [e_{\pm\alpha_i}, e_{\pm\alpha_k}]_q]_q = 0. \quad (5.4)$$

Here in (5.1)–(5.4) the brackets  $[\cdot, \cdot]$  is the usual supercommutator and  $[\cdot, \cdot]_q$  and  $\text{ad}_q$  denote the  $q$ -deformed supercommutator ( $q$ -supercommutator) in  $U_q(\mathfrak{g})$  [10]:

$$(\text{ad}_q e_\alpha) e_\beta \equiv [e_\alpha, e_\beta]_q = e_\alpha e_\beta - (-1)^{\deg(e_\alpha) \deg(e_\beta)} q^{(\alpha, \beta)} e_\beta e_\alpha, \quad (5.5)$$

where  $(\alpha, \beta)$  is a scalar product of the roots  $\alpha$  and  $\beta$ , and the parity function  $\deg(\cdot)$  is given by

$$\deg(k_{\alpha_i}) = 0 \quad (i \in I), \quad \deg(e_{\pm\alpha_i}) = 0 \quad (i \notin \tau), \quad \deg(e_{\pm\alpha_i}) = 1 \quad (i \in \tau). \quad (5.6)$$

Below we shall use the following short notation:

$$\vartheta(\gamma) := \deg(e_\gamma). \quad (5.7)$$

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<sup>8</sup>These (super)algebras include all finite-dimensional simple Lie algebras and basic classical superalgebras, infinite-dimensional affine Kac-Moody algebras [19] and superalgebras [20].

<sup>9</sup>What we really consider here is a special case when the system  $\Pi := \{\alpha_1, \dots, \alpha_r\}$  has a minimal number of odd roots. In the case if  $\Pi$  does not satisfies this condition then another additional Serre's relations can exist (see [22, 23]).

The definition of a quantum algebra also includes operations of a co-multiplication  $\Delta_q$ , an antipode  $S_q$ , and a co-unit  $\epsilon_q$ . Explicit formulas of these operations will not be used here and they are not given.

The  $q$ -analog of the Cartan-Weyl basis for  $U_q(\mathfrak{g})$  is constructed by using the following inductive algorithm [10, 11, 17], [24] – [26].

We fix some normal ordering  $\vec{\Delta}_+(\mathfrak{g})$  and put by induction

$$e_\gamma := [e_\alpha, e_\beta]_q, \quad e_{-\gamma} := [e_{-\beta}, e_{-\alpha}]_{q^{-1}} \quad (5.8)$$

if  $\gamma = \alpha + \beta$ ,  $\alpha \prec \gamma \prec \beta$  ( $\alpha, \beta, \gamma \in \vec{\Delta}_+(\mathfrak{g})$ ), and the segment  $[\alpha; \beta] \subseteq \vec{\Delta}_+(\mathfrak{g})$  is minimal one including the root  $\gamma$ , i.e. the segment has not another roots  $\alpha'$  and  $\beta'$  such that  $\alpha' + \beta' = \gamma$ . Moreover we put

$$k_\gamma := \prod_{i=1}^r k_{\alpha_i}^{l_i^{(\gamma)}}, \quad (5.9)$$

if  $\gamma = \sum_{i=1}^r l_i^{(\gamma)} \alpha_i$  ( $\gamma \in \vec{\Delta}_+(\mathfrak{g})$ ,  $\alpha_i \in \Pi$ ).

By this procedure one can construct the total quantum Cartan-Weyl basis for all quantized finite-dimensional simple contragredient Lie (super)algebras. In the case of the quantized infinite-dimensional affine Kac-Moody (super)algebras we have to apply one more additional condition. Namely, first we construct all root vectors  $e_\gamma$  ( $\gamma \in \vec{\Delta}_+(\mathfrak{g})$ ) by means of the given procedure, and then we overdeterminate the generators  $e_{\pm m\delta}$  of the imaginary roots  $m\delta \in \vec{\Delta}_+(\mathfrak{g})$  ( $m \in \mathbb{N}$ ) in a way that the new generators  $e'_{m\delta}$  are mutually commutative if they are not conjugate generators. Because of the fact that we do not have a sufficient place here to describe the overdetermination of imaginary root generators in details, we are restricted to a consideration of finite-dimensional case, i.e. when  $\mathfrak{g}$  is a finite-dimensional simple contragredient Lie (super)algebra.

The quantum Cartan-Weyl basis is characterized by the following properties [10, 11, 17], [24] – [28].

**Proposition 5.2** *The root vectors  $\{e_{\pm\gamma}\}$  ( $\gamma \in \vec{\Delta}_+(\mathfrak{g})$ ) satisfy the following relations:*

$$k_\alpha^{\pm 1} e_\gamma = q^{\pm(\alpha, \gamma)} e_\gamma k_\alpha^{\pm 1}, \quad (5.10)$$

$$[e_\gamma, e_{-\gamma}] = a(\gamma) \frac{k_\gamma - k_\gamma^{-1}}{q - q^{-1}}, \quad (5.11)$$

$$[e_\alpha, e_\beta]_q = \sum_{\alpha \prec \gamma_1 \prec \dots \prec \gamma_n \prec \beta; \{m_i\}} C_{\{m_i\}, \{\gamma_i\}} e_{\gamma_1}^{m_1} e_{\gamma_2}^{m_2} \cdots e_{\gamma_n}^{m_n}, \quad (5.12)$$

where  $\sum_i^n m_i \gamma_i = \alpha + \beta$ , and the coefficients  $C_{\dots}$  are rational functions of  $q$  and they do not depend on the Cartan elements  $k_{\alpha_i}$ ,  $i = 1, 2, \dots, r$ , and also

$$[e_\beta, e_{-\alpha}] = \sum C'_{\{m_i\}, \{\gamma_i\}; \{m'_j\}, \{\gamma'_j\}} e_{-\gamma_1}^{m_1} e_{-\gamma_2}^{m_2} \cdots e_{-\gamma_p}^{m_p} e_{\gamma'_1}^{m'_1} e_{\gamma'_2}^{m'_2} \cdots e_{\gamma'_s}^{m'_s} \quad (5.13)$$

where the sum is taken on  $\gamma_1, \dots, \gamma_p, \gamma'_1, \dots, \gamma'_s$  and  $m_1, \dots, m_p, m'_1, \dots, m'_s$  such that

$$\gamma_1 \prec \dots \prec \gamma_p \prec \alpha \prec \beta \prec \gamma'_1 \prec \dots \prec \gamma'_s, \quad \sum_l (m'_l \gamma'_l - m_l \gamma_l) = \beta - \alpha$$

and the coefficients  $C'_{\dots}$  are rational functions of  $q$  and  $k_\alpha$  or  $k_\beta$ . The monomials  $e_{\gamma_1}^{n_1} e_{\gamma_2}^{n_2} \cdots e_{\gamma_p}^{n_p}$  and  $e_{-\gamma_1}^{n_1} e_{-\gamma_2}^{n_2} \cdots e_{-\gamma_p}^{n_p}$ , ( $\gamma_1 \prec \gamma_2 \prec \dots \prec \gamma_p$ ), generate (as a linear space over  $U_q(\mathcal{H})$ ) subalgebras  $U_q(\mathfrak{b}_+)$  and  $U_q(\mathfrak{b}_-)$  correspondingly. The monomials

$$e_{-\gamma_1}^{n_1} e_{-\gamma_2}^{n_2} \cdots e_{-\gamma_p}^{n_p} e_{\gamma'_1}^{n'_1} e_{\gamma'_2}^{n'_2} \cdots e_{\gamma'_s}^{n'_s}, \quad (5.14)$$

where  $\gamma_1 \prec \gamma_2 \prec \dots \prec \gamma_p$  and  $\gamma'_1 \prec \gamma'_2 \prec \dots \prec \gamma'_s$ , generate  $U_q(\mathfrak{g})$  over  $U_q(\mathcal{H})$ .

Here the algebra  $U_q(\mathcal{H})$  is generated by the Cartan elements  $k_{\alpha_i}$  ( $i = 1, 2, \dots, r$ ).

Any formal Taylor series of  $TU_q(\mathfrak{g})$  is constructed from the monomials of the form (5.14) with coefficients, which are rational functions of the Cartan elements  $k_{\alpha_i}$  ( $i = 1, 2, \dots, r$ ), and nonnegative integers  $n_1, n_2, \dots, n_p, n'_1, n'_2, \dots, n'_s$  are subjected to the constraints of the type (4.9) for all monomial of the given series.

By definition, the extremal projector for  $U_q(\mathfrak{g})$  is a nonzero element  $p := p(U_q(\mathfrak{g}))$  of the Taylor extension  $TU_q(\mathfrak{g})$ ,  $p \in TU_q(\mathfrak{g})$ , satisfying the equations

$$e_{\alpha_i} p = p e_{-\alpha_i} = 0 \quad (\forall \alpha_i \in \Pi), \quad p^2 = p. \quad (5.15)$$

We fix some normal ordering  $\vec{\Delta}_+(\mathfrak{g})$  and let  $\{e_{\pm\gamma}\}$  ( $\gamma \in \underline{\Delta}_+(\mathfrak{g})$ ) be the corresponding Cartan-Weyl generators. The following statement holds for any quantized finite-dimensional contragredient Lie (super)algebra  $g$  [10, 11, 21]<sup>10</sup>.

**Theorem 5.3** *The equations (5.15) have a unique nonzero solution in the space of the Taylor extension  $TU_q(\mathfrak{g})$  and this solution has the form*

$$p = \prod_{\gamma \in \vec{\Delta}_+(\mathfrak{g})} p_\gamma, \quad (5.16)$$

where the order in the product coincides with the chosen normal ordering of  $\underline{\Delta}_+(\mathfrak{g})$  and the elements  $p_\gamma$  are defined by the formulae

$$p_\gamma = \sum_{m \geq 0} \frac{(-1)^m}{(m)_{\bar{q}_\gamma}!} \varphi_{\gamma,m} e_{-\gamma}^m e_\gamma^m, \quad (5.17)$$

$$\varphi_{\gamma,m} = \frac{(q - q^{-1})^m q^{-\frac{1}{4}m(m-3)(\gamma,\gamma)} q^{-m\rho(\gamma)}}{(a(\gamma))^m \prod_{l=1}^m \left( k_\gamma q^{\rho(\gamma)+\frac{l}{2}(\gamma,\gamma)} - (-1)^{(l-1)\vartheta(\gamma)} k_\gamma^{-1} q^{-\rho(\gamma)-\frac{l}{2}(\gamma,\gamma)} \right)}. \quad (5.18)$$

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<sup>10</sup>The theorem is also valid for the quantized infinite-dimensional affine Kac-Moody (super)-algebras, but in this case the formulas (5.17) and (5.18) for the imaginary roots  $\gamma = m\delta$  ( $m \in \mathbb{N}$ ) should be more detailed (see [11, 29] as examples).

Here  $\rho$  is the linear function such that  $\rho(\alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$  for all simple roots  $\alpha_i \in \Pi$ ;  $a(\gamma)$  is a factor in the relation (5.11);  $\bar{q}_\gamma := (-1)^{\vartheta(\gamma)} q^{-(\gamma, \gamma)}$ ; the symbol  $(m)_q!$  is given as follows:

$$(m)_q! = (m)_q(m-1)_q \cdots (1)_q, \quad (0)_q! = 1, \quad (m)_q := \frac{q^m - 1}{q - 1}. \quad (5.19)$$

In the limit  $q \rightarrow 1$  we obtain the extremal projector for the (super)algebra  $\mathfrak{g}$ :  $\lim_{q \rightarrow 1} p(U_q(\mathfrak{g})) = p(\mathfrak{g})$ .

A proof of the theorem actually reduces to the proof for the case of the quantized (super)algebras of rank 2, and it is similar to the case of non-deformed finite-dimensional simple Lie algebras [7, 13].

## 6 Bibliographic comments on applications of extremal projectors

For convenience of the reader, we present here the most important references related to the applications of extremal projectors.

- (i) An explicit description of irreducible representations of Lie (super)algebras (construction of various bases, action of generators, and their properties). Results (Gelfand–Tsetlin bases) are contained in the following sources: [30] for  $\mathfrak{su}(n)$ , V.N. Tolstoy (1975, unpublished) for  $\mathfrak{so}(n)$ , [31] for  $G_2$ , [32] for  $\mathfrak{sp}(2n)$ , [18] for  $\mathfrak{osp}(1|2)$ , [33] for  $\mathfrak{gl}(n|m)$ , [10, 21] for  $U_q(\mathfrak{su}(n))$ , [34] for  $U_q(\mathfrak{su}(n|1))$  and [35] for  $U_q(\mathfrak{su}(n, 1))$ .
- (ii) Theory of the Clebsch–Gordan coefficients of simple Lie algebras. Results can be found in [36]–[38] for  $\mathfrak{su}(3)$ , [39]–[42] for  $U_q(\mathfrak{su}(2))$ , [43, 44] for  $U_q(\mathfrak{su}(3))$  and [45] for  $U_q(\mathfrak{su}(n))$ .
- (iii) Description of reduction algebras (Mikelsons algebras). Results are given in [46] for  $A_n, B_n, C_n$ , and  $D_n$ ; [47] for  $\mathfrak{su}(n|m)$  and  $\mathfrak{osp}(m|2n)$ ; [35] for  $U_q(\mathfrak{gl}(n))$ .
- (iv) Description of Verma modules of Lie (super)algebras (singular vectors and their properties). Results for the simple Lie algebras can be found in [48].
- (v) Construction of solutions to the Yang–Baxter equation by using the projection operators. Results for  $\mathfrak{u}(3)$  and  $\mathfrak{u}(n)$  are given in [49, 50].
- (vi) Relation between extremal projection operators and integral projection operators. Results for simple Lie groups and algebras can be found in [51].
- (vii) Relation between extremal projection operators and canonical elements. Results for  $q$ -boson Kashiwara algebras were obtained in [52].
- (viii) Generalization of extremal projection operators. Results for  $\mathfrak{sl}(2)$  and  $U_q(\mathfrak{sl}(2))$  can be found in [47] and [54], respectively.
- (ix) Construction of indecomposable representations. Results for  $U_q(\mathfrak{sl}(2))$  are given in [54].

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